

First law of thermodynamics on holographic screens in entropic force frame

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By following the spirit of Verlinde's entropic force proposal arXiv:1001.0785, we give the differential and integral form of the first law of thermodynamics on the holographic screen enclosing a spherical symmetric black hole. It is consistent with equipartition principle and the form of Komar mass. The entropy of the holographic screen determines its area, i. e. $S = \frac{A}{4}$. And then we express the metric by thermodynamic variables, to give an illustration of how the space is foliated by the thermodynamical potentials.

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I. INTRODUCTION

One of the most interesting discovery in general relativity is the black hole thermodynamics, which gives an implication of the relation between gravity and thermodynamics [1]. Recently, it is wide believed that gravity might be originated from the thermodynamics of the unknown microstructure of spacetime. Jacobson first illustrates this idea by obtaining Einstein's equation from the first law of thermodynamics, $\delta Q = TdS$, which are defined on a local Rindler horizon. This illustration can be extended to non-Einstein gravity (for a review, see [3]). An alternative illustration is the entropic force proposed by Verlinde [4]. In his work, spacetime is the place where information is stored, and the fundamental unit of 3-dimensional space is 2-dimensional holographic screen. By the assumption of thermodynamical properties on the holographic screens and the principle of holography, Verlinde gives the correspondence between gravity in 3-dimensional space and thermodynamics on the 2-dimensional holographic screen. Namely, the origin of the gravity is the entropic force. Entropic force soon gets various applications and comments, for an incomplete list, see [5].

In Verlinde's proposal, each holographic screen is assigned with a temperature. The Komar energy of the system enclosed by the screen is evenly divided over the degrees of freedom of holographic screen according to the temperature. And gravity is originated by the entropy variation when a test particle is approaching the screen. However, Verlinde doesn't give the form of the thermodynamical entropy. In the work of Padmanabhan [6], Komar energy can be expressed as equipartition principle, $E = 1/2 \int TdN$. For Einstein theory, we have $dN \sim dA$, and A is the area of the holographic screen. And for a bifurcation horizon, he gets $E = 2TS$, where S is the entropy on the horizon. Thus we can guess the entropy of the holographic screen may be $S = A/4$, as it is on the horizon. In [7–9, 12], the authors also get the relation

$S = A/4$ from different aspects in the idea of entropic force, and S is the entropy on holographic screen. Because T , E and S are well defined on holographic screen, it is necessary for us to study the first law of the holographic screen thermodynamics.

The black hole horizon is a natural choice of holographic screen, so it is reasonable to assume that holographic screen thermodynamics are similar to black hole thermodynamics. Besides, in [10–12], it is found that when a holographic screen coincides with the black horizon, the definition of Komar energy is equivalent to the Smarr law, which is the integrated form of first law of black hole thermodynamics. As a result, when the holographic screen is far from the horizon, it is reasonable to assume that the equivalence is still valid. Since Smarr law is obtained by integrating the differential form of the first law of black hole thermodynamics, it is required that there must be a differential first law of holographic screen thermodynamics.

Moreover, both in Jacobson and Verlinde's work, Unruh temperature detected by an accelerated observer in each spacetime point plays a key role. It appears in the first law $\delta Q = TdS$ on the Rindler horizon and it is equivalent to the holographic screen temperature. Thus, it is necessary for a corresponding first law of thermodynamics on the holographic screen. In [13], Zhao discusses the Poincaré symmetry of the first law of thermodynamics. It indicates that thermodynamics as well as gravity is universal for all the physical system. This is another implication of the existence of first law of holographic thermodynamics.

In this article, by using the similar method of Smarr, we give the first law on the holographic screen enclosing a spherical black hole, in both the differential and integrated form. The integrated form is consistent to the energy equipartition principle and the Komar mass energy. We find that the screen entropy still has the relation $S \sim A/4$, but we reverse the logic of the interpretation to $A \sim 4S$. That is, the area of the holographic screen is determined by its entropy. Comparing to Newton potential, the holographic screen thermal potential, such as temperature, is more appropriate for foliating the space. As a result, not only the Einstein equation is an equation of thermodynamical state [2], but also the space metric

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can be written as functions of thermodynamical entities. This a complementary illustration of the emergency of spacetime by Verlinde.

In section II, we start with a short review of the definition of temperature and energy from entropic force. The first law of holographic screen thermodynamics is given in section III. In section IV, we illustrate our method on an RN black hole. Then we discuss the physical meaning of $S = \frac{A}{4}$ and how the thermodynamics generates the space. The conclusion are presented in section V.

II. TEMPERATURE AND ENERGY ON HOLOGRAPHIC SCREEN

In this section, we review the definition of temperature and energy on the holographic screen by Verlinde [4]. In the context we use the units $G = \hbar = c = k_B = 1$.

From the perspective of Verlinde, the three-dimensional theory of gravity as well as the space are originated from the thermodynamics of the two-dimensional holographic screens covering the three-dimensional space. The holographic screens are characterized by temperature and energy, and the ‘number of bits’ from the unknown microstructure of spacetime.

In the spacetime with a global time-like Killing vector field ξ , we can define the Newton potential

$$\phi = \frac{1}{2} \ln(-\xi^2). \quad (1)$$

The holographic screen is defined as an equipotential surface. The four acceleration of a particle close to the holographic screen is given by

$$a^\mu = -\nabla^\mu \phi. \quad (2)$$

Then the holographic screen temperature is defined as the Unruh temperature for the acceleration,

$$T = -\frac{1}{2\pi} e^\phi n^\mu a_\mu = \frac{1}{2\pi} e^\phi \sqrt{\nabla^\mu \phi \nabla_\mu \phi}. \quad (3)$$

Here, $n^\mu = \nabla^\mu \phi / \sqrt{\nabla^\nu \phi \nabla_\nu \phi}$ is the normal vector on the screen. The energy of the screen is defined by the Komar mass energy [10, 12],

$$E_{Komar}(\mathcal{S}, \xi) = \frac{1}{8\pi} \int_{\mathcal{S}} *d\xi = \frac{1}{2} \int_{\mathcal{S}} T dN. \quad (4)$$

Here, $N = A$ is the number of bits stored on the holographic screen, assumed by Verlinde [4], and \mathcal{S} denotes a holographic screen. The last equation in (4) can be interpreted as energy equipartition rule on the screen.

III. THE FIRST LAW ON HOLOGRAPHIC SCREEN

In this section, we follow the similar method of Smarr to obtain the formula of the first law of thermodynamics

on holographic screen in 3+1 dimensions, which enclosing a spherical symmetric black hole. The general metric of this spacetime takes the form,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad (5)$$

where $d\Omega$ is the line element of an unit 2-sphere. If the spacetime is asymptotically flat, the function $f(r)$ satisfies,

$$\lim_{r \rightarrow \infty} f(r) = 1. \quad (6)$$

When we reach the event horizon r_0 . we have,

$$f(r)|_{r=r_0} = 0. \quad (7)$$

As a result, we get the Killing vector, Newton potential, and Unruh-Verlinde temperature expressed as [12],

$$\xi_\mu = (-f(r), 0, 0, 0), \quad (8)$$

$$\phi = \frac{1}{2} \ln f(r), \quad (9)$$

$$T = \frac{1}{4\pi} |f'(r)|. \quad (10)$$

Before talking about the thermodynamics, it should be assumed that the holographic screens are in a thermal equilibrium. Thus, the holographic screen is at least an isothermal surface, i. e. $f'(r) = \text{const}$. The Newton potential ϕ is also a constant on the isothermal surface, so we begin with a holographic screen determined by,

$$f(r) = e^{2\phi} = c. \quad (11)$$

Here, c is a constant, ranging in $[0, 1]$. In [4], Verlinde asserts that the amount of coarse graining for the information on the screens is measured by $-\frac{\phi}{2} = -\frac{\ln c}{4}$. For a black hole, $f(r)$ is also a function of mass and other thermal entities. Thus, the above equation can also be expressed as,

$$f(r, M, Q_1, \dots, Q_n) = c. \quad (12)$$

Here, $\{Q_n\}$ are n conserved charges for the spherical black hole. Following the similar tricks of Smarr [14], we solve M as,

$$M = \mathcal{M}(c, r, Q_1, \dots, Q_n). \quad (13)$$

Differentiating the equation, we get,

$$\begin{aligned} dM &= \frac{\partial \mathcal{M}(c, r, Q_1, \dots, Q_n)}{\partial r} dr + \sum_i \frac{\partial \mathcal{M}(c, r, Q_1, \dots, Q_n)}{\partial Q_i} dQ_i \\ &= \frac{\partial \mathcal{M}(c, r, Q_1, \dots, Q_n)}{\partial r} \frac{\partial r}{\partial \mathcal{A}} d\mathcal{A} + \sum_i \frac{\partial \mathcal{M}(c, r, Q_1, \dots, Q_n)}{\partial Q_i} dQ_i. \end{aligned} \quad (15)$$

Here, \mathcal{A} is an undetermined area-like function. Eq. (15) can be perceived as the extended first law of black hole thermodynamics obtained by Smarr [14], $dM = T d\mathcal{A} +$

$\Phi dQ + \dots$, iff we identify the coefficient of $d\mathcal{A}$ in Eq. (15) with T determined in Eq. (10). So,

$$\frac{\partial \mathcal{M}}{\partial r} \frac{\partial r}{\partial \mathcal{A}} = \frac{\partial f}{\partial r} \left(\frac{\partial f}{\partial M} \right)^{-1} \frac{\partial r}{\partial \mathcal{A}} = \frac{1}{4\pi} f'(r) \quad (16)$$

and,

$$\mathcal{A} = -4\pi \int \left(\frac{\partial f}{\partial M} \right)^{-1} dr. \quad (17)$$

The area-like function \mathcal{A} is determined by $f(r, M, Q_1, \dots, Q_n)$. Because the derivative $\frac{\partial f}{\partial M}$ is determined by the equation of motion, \mathcal{A} is also determined by the equation of motion, and it varies in different theories of gravity. In general relativity, for a spherical black hole, we already know that, $f(r, M, Q_1, \dots, Q_n) = 1 - \frac{2M}{r} + g(r, Q_1, \dots, Q_n)$. So, in this case, $\mathcal{A} = \pi r^2 = \frac{A}{4}$, which reproduces the black hole entropy when $c = 0$ [16].

Because the metric is spherical symmetric, the other thermodynamical potentials $\{\Phi_i = \frac{\partial \mathcal{M}}{\partial Q_i}\}$ are also spherical symmetric, and are constant on the holographic screen. So, it is reasonable for us to treat the holographic screen as in a thermal equilibrium. And the first law of holographic screen is obtained by rewriting Eq. (15) as,

$$dM = T d\mathcal{A} + \sum_i \Phi_i dQ_i. \quad (18)$$

It is our key conclusion in this section. This formula looks like the black hole thermodynamics. However, Eq. (18) is defined on any holographic screen with $0 \leq c \leq 1$, while the black hole thermodynamics is only the case that $c = 0$. The first law of holographic screen thermodynamics describes the variation between two adjacent thermal states of the screens, or the variation after an infinitesimal heat flow absorbed by the screens. So, Eq. (18) is also consistent with Jacobson's work [2]. It indicates that the gravity is not only a dual of the first law on a local Rindler horizon, which is a null-like surface, but also a dual of thermodynamics on a holographic screen which is not null-like.

IV. RN BLACK HOLE AND INTEGRATED FIRST LAW

A simple case of the above analysis is the screens enclosing an RN black hole, with an electric charge Q in metric (5),

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \quad (19)$$

Setting $f(r) = c$, we will get two solutions of r , and we only take the solution of the screens which are always lying outside the event horizon,

$$r_s = \frac{M + \sqrt{M^2 - (1-c)Q^2}}{1-c}. \quad (20)$$

Solving M in Eq. (20)

$$M = \frac{r_s}{2} \left(1 - c + \frac{Q^2}{r_s^2} \right), \quad (21)$$

And substituting into Eq. (18), we have,

$$dM = T d\mathcal{A} + \Phi dQ. \quad (22)$$

Here $T = \frac{f'(r_s)}{4\pi} = \frac{Mr_s - Q^2}{2\pi r_s^3} = \frac{(1-c)r_s^2 - Q^2}{4\pi r_s^3}$, $\mathcal{A} = \pi r_s^2$, and $\Phi = \frac{Q}{r_s}$. When $c = 0$, it is the first law of thermodynamics of RN black hole. To obtain the integrated form of first law, we integrate Eq. (22) by a path in the phase space of the holographic screen,

$$M = \int_{(r=0, Q'=0)}^{(r=r_s, Q'=0)} T d\mathcal{A} + \int_{(r=r_s, Q'=0)}^{(r=r_s, Q'=Q)} \Phi dQ'. \quad (23)$$

The first integration gets $2T\mathcal{A} + \frac{1}{2}\Phi Q$, and the second integration gets $\frac{1}{2}\Phi Q$. As a result,

$$M = 2T\mathcal{A} + \Phi Q. \quad (24)$$

This is the integrated form of the first law on the holographic screen, which is equivalent to the Komar mass energy and the equipartition principle,

$$M - \Phi Q = 2T\mathcal{A} = \frac{1}{2}NT = \frac{1}{8\pi} \int_S *d\xi = E_{Komar}(\mathcal{S}, \xi). \quad (25)$$

So far we have illustrated the differential and integrated form of the first law of holographic screen thermodynamics, with the screen enclosing an RN black hole. We can see that \mathcal{A} in the first law plays the same role as the entropy on the holographic screen, i. e. $S = \mathcal{A} = \frac{A}{4}$. It is consistent with the black hole entropy and the holographic screen entropy obtained by different method in [7–9, 12]. This interpretation of \mathcal{A} might face two problems. Why it breaks the Bekenstein entropy bound $S \leq 2Er$ [4]? When a heat flow $\Delta Q = T\Delta S$ is absorbed by the holographic screen, will the total entropy $S + \Delta S$ exceed a quarter of the area πr^2 ?

The answer for the first problem is that the entropy $S = \mathcal{A} = \frac{A}{4}$ here is defined on the holographic screen, just like the equipartition energy $E = M - \Phi Q$, and the temperature T . The entropy S in Bekenstein bound is defined in the bulk, which is not applied on the holographic screen. The thermal entities defined on the holographic screen should be the dual of the gravitational quantities in the bulk. For example, the energy on the screen is the dual of the gravitational energy in the bulk, the temperature is the dual of the gravitational acceleration, and what is the dual of the entropy on the screen? It is the area of the 2-dimensional boundary of the gravitational system.

Also, we have answered the second problem at the same time. Once a heat flow is absorbed by the screen, the information stored on the screen (or in the bulk respectively) is increased, and the 2-dimensional space on

the screen is enlarged by the same rate. This is how the space is emergent from the thermodynamics of the unknown microstructure. When there is no information, there is no space, and there is no matter either. When the information comes up, the 2-dimensional space is spanned by the amount of the thermodynamical entropy of the information. Then the amount of matter is increased by the heat flow $\Delta Q = T\Delta S$. The temperatures on different holographic screens can be different, so $T_1\Delta S_1 = T_2\Delta S_2$. Numerically, we have $T_1r_1\Delta r_1 = T_2r_2\Delta r_2$, where we have defined $r = \sqrt{S/\pi}$. Thus, in the view of thermodynamics on the screen, there is a third dimension emergent from different screens with the distance $(\Delta r_1 - \Delta r_2)$. While in the view of gravity in the bulk, the holographic screens expand to new positions with new areas as the black hole horizon does in the same process.

Numerically, the above integrated form of first law Eq. (25) is the same to Eq. (21), because we have just differentiated it and then integrated it. Notice that Eq. (21) is obtained by $f(r) = c$, which is a component of spacetime metric in general relativity without considering thermodynamics. It indicates that there is a deep relationship between gravity and holographic screen thermodynamics. i. e. thermodynamics is hidden in gravity, or gravity is a reflection of the holographic screen thermodynamics. In [2], Jacobson argued that Einstein equation “is born in the thermodynamic limit as a relation between thermodynamic variables”. So, it is straightforward to see that, the spacetime metric which is solution to Einstein equation, should have a counterpart in thermodynamics.

Let us realized the argument specifically. Since $S = \frac{A}{4}$, the spherical 2-dimensional part of the metric can write as

$$r^2 d\Omega^2 = \frac{S}{\pi} d\Omega^2. \quad (26)$$

Thus, to integrate the spherical part of the metric is to run over all degrees of freedom on the holographic screen. However, there is a freedom of rewriting the radial part of the metric. From $f(r) = c$ and $T = \frac{(1-c)r_s^2 - Q^2}{4\pi r_s^3}$, we have,

$$\frac{1}{f(r)} dr^2 = \frac{1}{1 - \frac{4TS + \Phi Q}{\sqrt{S/\pi}}} (d\sqrt{S/\pi})^2 = \frac{1}{1 - \frac{2M - \Phi Q}{\sqrt{S/\pi}}} (d\sqrt{S/\pi})^2. \quad (27)$$

Here dS means the variation of the entropy between two screens in adjacent states. It shows the way that how the space is foliated by the thermodynamical potential. If we set $\Phi = 0$, it is the radical metric outside a Schwarzschild black hole. If we set $\Phi = 0$ and $T = 0$, it is the radical metric of a flat space. If we define a non zero potential Φ' and a charge Λ , we can also get the metric of an AdS RN black hole. So viewed in thermodynamics, the space is foliated by the respective thermodynamical potentials other than Newton potential ϕ .

V. CONCLUSION AND DISCUSSIONS

The differential and integrated form first law of holographic thermodynamics are obtained from the gravity of spherical black hole. And the entropy of the holographic screen is $S = \frac{A}{4}$, with the meaning that the area is determined by the amount of entropy. By expressing the spherical solution of Einstein equation by thermodynamic variables, we give an illustration of how the 3-dimensional space is emergent from the thermodynamics. It indicates that the solution (metric) as well as Einstein equation are born in thermodynamics.

The radical metric of Eq. (27) suffers a drawback from the arbitrariness of rewriting $f(r)$ as thermodynamical entities. e. g. Q^2/r^2 can also write as Φ^2 . However, we should notice that the equivalency between the first law of thermodynamics and the equation of motion of gravity. So, when we put $f(r)$ in the equation of motion, it should be equivalent to the first law. Thus we can not arbitrarily express $f(r)$ by thermodynamical entities, and we hope we will find a way to fix it carefully in future.

The work is discussed in the spherical black hole case, in which the thermodynamical potentials are found to be constant on the screen. If we want to extend this work on axisymmetric case, such as Kerr-Newman black holes, we will find that the temperature, the electric potential, the angular velocity and the Newton potential are constant only on the black hole horizon [12]. It means the holographic screens outside the black hole horizon may not be states of thermal equilibrium. The dual of thermodynamics and gravity is hidden deeper in this case.

Note added: When we are in the final stage of writing the manuscript, a paper [15] appears in the preprint archive, which discusses some relevant topics with our discussion in this paper. By using geometrical method, the author extends the result of Jacobson to a time-like screen of observers of finite acceleration. The form of his result is significantly different from ours, and the relation between his result and our result still needs to be clarified.

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